CS 163 Discrete Math **Fall 2023**
http://neilklingensmith.com/teaching/loyola/cs163/

Final Exam

Date: December 6, 2023

Name:

1. Gram-Schmidt Orthogonalization.

   (a) (10 points) Use Gram-Schmidt Orthogonalization to orthogonalize the following matrix A by hand. Show your work on all parts.

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix}$$

Solution:

$$u_1 = v_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$u_2 = v_2 - \frac{< v_2, u_1 >}{< u_1, u_1 >} u_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \left(\frac{-6}{8}\right) \begin{bmatrix} 2 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix}$$

$$e_1 = \frac{u_1}{||u_1||} = \frac{1}{\sqrt{4+4}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$e_2 = \frac{u_2}{||u_2||} = \frac{1}{\sqrt{50/4}} \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix} = \frac{1}{\frac{5}{2}\sqrt{2}} \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

   (b) (5 points) Show that Q is orthogonal by computing $Q^T Q$

Solution:

$$\mathbf{Q}\mathbf{Q}^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   (c) (10 points) Find the QR factorization of matrix A.

Solution:

$$\mathbf{A} = \mathbf{QR}$$

$$\mathbf{R} = \mathbf{Q}^T\mathbf{A} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -3/\sqrt{2} \\ -4/\sqrt{2} & 3/\sqrt{2} \end{bmatrix}$$

(d) (5 points) Show that $QR = A$ by multiplying Q by R.

Solution:

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & -3/\sqrt{2} \\ -4/\sqrt{2} & 3/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix}$$

(e) (10 points) Solve the following system for $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ *Hint: substitute* $\mathbf{A} = \mathbf{QR}$, *then multiply both sides of the equation by* $\mathbf{Q}^T$. *Then multiply* $\mathbf{R}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ *and solve for* x *and* y.

$$\begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3/\sqrt{2} \\ 0 \end{bmatrix}$$

Solution:

$$\mathbf{A}x = b$$

$$\mathbf{QR}x = b$$

$$\mathbf{R}x = \mathbf{Q}^T b$$

$$\begin{bmatrix} 0 & -3/\sqrt{2} \\ -4/\sqrt{2} & 3/\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -3/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -3/2 \\ -3/2 \end{bmatrix}$$

$$\frac{-3}{\sqrt{2}} y = -3/2$$

$$y = 2\sqrt{2}$$

$$\frac{-4}{\sqrt{2}} x + \frac{3}{\sqrt{2}} y = \frac{-4}{\sqrt{2}} x + \frac{3}{\sqrt{2}} 2\sqrt{2} = \frac{-4}{\sqrt{2}} x + 6 = -3/2$$

$$\frac{-4}{\sqrt{2}} x = -15/2$$

$$x = \frac{15}{8}\sqrt{2}$$

2. **Filtering a Signal from Noise** You are measuring a signal that is corrupted by noise:

$$m = s + n$$

Where s is the true signal you are trying to measure, n is the noise, and m is your measurement. All three (m, s, and n) are 2-dimensional vectors, and the noise n is smaller in magnitude than the signal s. Also, n is orthogonal to s.

You have a 2×2 orthogonal matrix $\mathbf{Q}$ where one column is the s dimension and the other column is the n dimension:

$$\mathbf{Q} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$$

Suppose you take the measurement $m = \begin{bmatrix} 9\sqrt{3}/16 \\ 5/16 \end{bmatrix}$

(a) (10 points) To separate the signal from the noise, compute the product $\mathbf{Q}m$.

Solution:

$$\mathbf{Q}m = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 9\sqrt{3}/16 \\ 5/16 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3}/8 \end{bmatrix}$$

(b) (5 points) The product from part (a) above is the representation of m under basis $\mathbf{Q}$. In the $\mathbf{Q}$ basis, one component of m is the signal only, and the other component is the noise only. Modify your vector, setting the noise component to zero and keeping the signal component the same as it was. *Hint: the signal component should be larger in magnitude than the noise.*

Solution:

$$r = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(c) (10 points) Now that you have eliminated the noise, rotate your modified vector from part (b) back to the standard basis by multiplying it by $\mathbf{Q}^T$:

Solution:

$$s = \mathbf{Q}^T r = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

(d) (10 points) An orthogonal projector $\mathbf{P}$ is a matrix that projects a vector into a subspace—kind of like what you just did above. Suppose you want to construct a projector that projects vectors into the subspace spanned by a vector w. The formula for the matrix that does this is $\mathbf{P} = ww^T$. Find a projector matrix $\mathbf{P}_s$ that projects the measurement signal m into the subspace spanned by s. *Hint: that subspace is one of the columns of the* $\mathbf{Q}$ *matrix.*

Solution:

$$w = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

$$\mathbf{P}_s = ww^T = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{bmatrix}$$

(e) (5 points) Show that $\mathbf{P}_s m = s$

Question	Points	Score
1	40	
2	40	
Total:	80	